

Nonlinear Wave Propagation - a fast 3D Simulation Method based on Quasi-linear Approximation of the Second Harmonic Field

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03.06.02, rev. 16.02.05

Abstract

Nonlinear wave propagation plays an important role in medical ultrasound imaging. By utilizing the second harmonic wave, a mayor improvement in image quality is achieved.

In order to optimize image quality, detailed knowledge of the second harmonic wave-field as a function of transducer geometry, focusing, and pulse shape is needed.

Numerical simulation of nonlinear wave propagation for pulsed wave ultrasound is time consuming, especially when the aperture is not circular symmetrical. By using the square of the linear propagating wave-field as a volume-source, a good approximation of the second harmonic wave can be achieved.

In our simulation method, we started with the spatial/time Fourier transform of the pulsed linear wave in the focal plane. By using the angular spectrum method, the contribution from an arbitrarily plane was found by back propagation from the focal plane, squaring the pressure wave, followed by forward propagation of the result to the observation plane. In the spatial/time Fourier-domain, propagation is performed by a complex exponential factor, whereas the square operation corresponds to a 3D convolution. It turned out that the integration along the ultrasound beam could be solved analytically. The computation in the Fourier-domain was then reduced to a trippel-integral. The resulting second harmonic wave in the observation plane was found by (3D)inverse FFT.

A simulation program has been implemented in Matlab, using a laptop PC with 1.2 GHz Pentium III processor. Applied to a phased array transducer with rectangular aperture 13*18mm and transmit frequency 1.7 MHz, the simulation time was less than 5 minutes.

Theory

The linear propagating pulsed pressure wave from a transducer with center in origo, is described by $p(x,y,z,t)$, where x,y,z are the azimuth, elevation, and propagation direction. We will apply the angular spectrum method to propagate the wave along the z -axis. For any point z_0 along the ultrasound beam, we can apply 3D Fouriertransform in the variables x,y , and t :

$$P(fx, fy, f | z_0) = \iiint dx dy dt p(x, y, z_0, t) \exp\left\{-i \frac{2\pi}{c} (fx \cdot x + fy \cdot y + cf \cdot t)\right\} \quad (1)$$

Note that the spatial frequency variables fx and fy are scaled by the speed of sound c , and have unit [Hz]. Using the angular spectrum method, the relation between the wave at two planes at z_1, z_2 along the z -axis is given by

$$P(fx, fy, f | z_2) = P(fx, fy, f | z_1) \exp\left\{-i \frac{2\pi}{c} (z_2 - z_1) \sqrt{f^2 - fx^2 - fy^2}\right\} \quad (2)$$

We will use vector notation for the three frequency variables:

$$P(\vec{f} | z_2) = P(\vec{f} | z_1) \exp\left\{-i K(\vec{f})(z_2 - z_1)\right\} \quad (3)$$

$$K(\vec{f}) \equiv \frac{2\pi}{c} \sqrt{f^2 - fx^2 - fy^2}; \text{ where } \vec{f} \equiv (fx, fy, f)$$

In the quasi-linear approximation to the second harmonic field, the square of the linear pressure field $p_1(x,y,z,t)$ acts as a volume source. The contribution to the second harmonic wave in position z_2 from the linear wave in position z , is obtained by squaring the linear wave, and propagate the result from z to z_2 . In the Fourier-domain, this corresponds to an autoconvolution in (fx, fy, f)

$$P_2'(\vec{f} | z_2; z) = P_1 \otimes P_1(\vec{f} | z) \exp\left\{-i K(\vec{f})(z_2 - z)\right\} \quad (4)$$

The complete second harmonic field is obtained by integrating (4) from $z=0$ (transducer surface) to $z=z_2$.

$$\begin{aligned}
P2(\vec{f} | z2) &= \int_0^{z2} dz P1 \otimes P1(\vec{f} | z) \exp\{-i K(\vec{f})(z2 - z)\} \\
&= \int_0^{z2} dz \iiint d\vec{f}' P1(\vec{f}' - \vec{f} | z) P1(\vec{f}' | z) \cdot \exp\{-i K(\vec{f})(z2 - z)\} \quad (5)
\end{aligned}$$

The linear field in any point z can be obtained from the field in one specific point $z1$ (e.g. the focal point), using equation (2)

$$P2(\vec{f} | z2) = \int_0^{z2} dz \iiint d^3 \vec{f}' P1(\vec{f} - \vec{f}' | z1) P1(\vec{f}' | z1) \exp\{-i [K(\vec{f} - \vec{f}') + K(\vec{f}')] (z - z1)\} \exp\{-i K(\vec{f})(z2 - z)\} \quad (6)$$

By changing the order of integration in (6), the second harmonic wave can be expressed as a trippel integral similar to a autoconvolution of the fundamental wave P1, with a kernel Hp which is independent of P1.

$$\begin{aligned}
P2(\vec{f} | z2) &= \iiint d^3 \vec{f}' P1(\vec{f} - \vec{f}' | z1) P1(\vec{f}' | z1) Hp(\vec{f}, \vec{f}') \\
Hp(\vec{f}, \vec{f}') &\equiv \int_0^{z2} dz \exp\{-i K(\vec{f})(z2 - z)\} \exp\{-i [K(\vec{f} - \vec{f}') + K(\vec{f}')] (z - z1)\} \quad (7)
\end{aligned}$$

Further development of the kernel Hp:

$$\begin{aligned}
Hp(\vec{f}, \vec{f}') &= \exp\{-i [z2 K(\vec{f}) - z1 K(\vec{f}') - z1 K(\vec{f} - \vec{f}')] \} \cdot \int_0^{z2} dz \exp\{-i [-K(\vec{f}) + K(\vec{f}') + K(\vec{f} - \vec{f}')] z\} \\
&= \exp(-i K(\vec{f})(z2 - z1)) \cdot \exp(i \Lambda(\vec{f}, \vec{f}') z1) \cdot \int_0^{z2} dz \exp\{-i \Lambda(\vec{f}, \vec{f}') z\} \\
\Lambda(\vec{f}, \vec{f}') &\equiv -K(\vec{f}) + K(\vec{f}') + K(\vec{f} - \vec{f}') \quad (8)
\end{aligned}$$

The integration over z is solved

$$\int_0^{z2} dz \exp\{-i \Lambda(\vec{f}, \vec{f}') z\} = i \frac{\exp(-i \Lambda(\vec{f}, \vec{f}') z2) - 1}{\Lambda(\vec{f}, \vec{f}')} = z2 \cdot \exp\left(-i \Lambda(\vec{f}, \vec{f}') \frac{z2}{2}\right) \cdot \text{sinc}\left(-\Lambda(\vec{f}, \vec{f}') \frac{z2}{2\pi}\right) \quad (9)$$

and the kernel takes on its final form

$$Hp(\vec{f}, \vec{f}') = z2 \cdot \exp\{-i K(\vec{f})(z2 - z1)\} \cdot \exp\left\{i \Lambda(\vec{f}, \vec{f}') \left(z1 - \frac{z2}{2}\right)\right\} \cdot \text{sinc}\left(-\Lambda(\vec{f}, \vec{f}') \frac{z2}{2\pi}\right) \quad (10)$$

Summary of equation (1)-(10): The second harmonic field P2 in observation plane position $z2$ can be derived from the fundamental field P1 in plane position $z1$.

$$\begin{aligned}
P2(\vec{f} | zo) &= \iiint d^3 \vec{f}' P1(\vec{f} - \vec{f}' | z1) P1(\vec{f}' | z1) Hp(\vec{f}, \vec{f}') \\
Hp(\vec{f}, \vec{f}') &\equiv z2 \cdot \exp(-i K(\vec{f})(z2 - z1)) \cdot \exp\left\{i \Lambda(\vec{f}, \vec{f}') \left(z1 - \frac{z2}{2}\right)\right\} \cdot \text{sinc}\left(-\Lambda(\vec{f}, \vec{f}') \frac{z2}{2\pi}\right) \\
&= z2 \cdot \exp\left(-i K(\vec{f}) \frac{z2}{2}\right) \cdot \exp\left\{i \left\{K(\vec{f}') + K(\vec{f} - \vec{f}')\right\} \left(z1 - \frac{z2}{2}\right)\right\} \cdot \text{sinc}\left(-\left\{-K(\vec{f}) + K(\vec{f}') + K(\vec{f} - \vec{f}')\right\} \frac{z2}{2\pi}\right) \\
\Lambda(\vec{f}, \vec{f}') &\equiv -K(\vec{f}) + K(\vec{f}') + K(\vec{f} - \vec{f}') \\
K(\vec{f}) &\equiv \frac{2\pi}{c} \sqrt{f^2 - fx^2 - fy^2}; \\
\vec{f} &\equiv (fx, fy, f) \quad (11)
\end{aligned}$$

When the observation plane is equal to the focal plane; $z2=z1$, the kernel takes on a simpler form

$$Hp(\vec{f}, \vec{f}') \equiv z2 \cdot \exp\left(i \Lambda(\vec{f}, \vec{f}') \frac{z2}{2}\right) \cdot \text{sinc}\left(-\Lambda(\vec{f}, \vec{f}') \frac{z2}{2\pi}\right) \quad (12)$$

Linear wave propagation to the focal plane

The computation time for numerical evaluation of equation (11) depends to a large extent on the spatial bandwidth of the fundamental wave in plane position $z=z_1$. Although any position z_1 along the beam may be used, the position of the focal plane seems to be the preferred choice.

In the focal plane of a focused transducer, the spatial Fourier transform of the wave equals the aperture function. This approximation is valid for large f-number, and describes the field on a spherical surface, with center in the transducer.

$$P1'(fx, fy, f) = P(f)A(R \frac{fx}{f}, R \frac{fy}{f}) \quad (13)$$

The focal distance R is equal in azimuth (x) and elevation (y), and A(x,y) is the aperture function.

To apply equation (11), the fundamental field P1 in a plane surface perpendicular to the ultrasound beam must be known. If we are looking at a limited opening angle, the pulsed wave in the focal plane can be obtained by a time-shift $\tau(x,y)$ of the wave corresponding to the propagation from the spherical surface, down to the focal plane

$$p1(x, y, t) = p1'(x, y, t - \tau(x, y)) \quad (14)$$

$$\tau(x, y) = \frac{1}{c} \sqrt{R^2 - x^2 - y^2}$$

The temporal Fourier transform of (14) gives

$$P1_f(x, y, f) = P1_f'(x, y, f) \exp\left\{i \frac{2\pi}{c} f \sqrt{R^2 - x^2 - y^2}\right\} \quad (15)$$

Spatial Fourier transform of (15) gives a convolution in (fx,fy)

$$P1(x, y, f) = P1' \otimes_{fx, fy} Hs(fx, fy, f) \quad (16)$$

$$Hs(fx, fy, f) \equiv \iint dx dy \exp\left\{i \frac{2\pi}{c} f \sqrt{R^2 - x^2 - y^2}\right\} \exp\left\{-i \frac{2\pi}{c} (fx \cdot x + fy \cdot y)\right\}$$

Summary of equation (13) – (16): The spatio-temporal Fourier transform in the focal plane $z=R$ of an aperture A(x,y) transmitting a pulse P(f) is given by

$$P1(x, y, f) = P1' \otimes_{fx, fy} Hs(fx, fy, f) \quad (17)$$

$$Hs(fx, fy, f) \equiv \text{Fourier}_{x,y} \left(\exp\left\{i \frac{2\pi}{c} f \sqrt{R^2 - x^2 - y^2}\right\} \right)$$

$$P1'(fx, fy, f) \equiv P(f)A(R \frac{fx}{f}, R \frac{fy}{f})$$

One limitation with this method is that the focal distance R is equal in azimuth and elevation. For conventional 1D arrays with fixed elevation focus, this will not in general be the case. However, for 1.5 D or 2D arrays, the focal distance will in most cases be equal, unless one want to extend the focal depth by using different focal distances in x and y.

The situation with different azimuth and elevation focus can be solved by using a complex (and frequency dependent) aperture function. The deviation from a spherical shape of the transducer surface can be compensated by a frequency and space dependent phase shift in the aperture function

$$A'(x, y, f) = A(x, y) \exp\left(-i \frac{2\pi f}{c} \Delta(x, y)\right) \quad (18)$$

$$\Delta(x, y) \equiv R - \sqrt{R^2 - x^2 - y^2} - R_x + \sqrt{R_x^2 - x^2} - R_y + \sqrt{R_y^2 - y^2}$$

Combining (17) and (18) gives

$$P1'(f_x, f_y, f) = P(f)A(R \frac{f_x}{f}, R \frac{f_y}{f}) \exp\left(-i \frac{2\pi f}{c} \Delta(R \frac{f_x}{f}, R \frac{f_y}{f})\right) \quad (19)$$

$$\Delta(R \frac{f_x}{f}, R \frac{f_y}{f}) = R - R \sqrt{1 - \left(\frac{f_x}{f}\right)^2 - \left(\frac{f_y}{f}\right)^2} - R_x + R_x \sqrt{1 - \left(\frac{R}{R_x}\right)^2 \left(\frac{f_x}{f}\right)^2} - R_y + R_y \sqrt{1 - \left(\frac{R}{R_y}\right)^2 \left(\frac{f_y}{f}\right)^2}$$

Effect of frequency dependent attenuation

The propagation equation (2) can be modified to include effects of frequency dependent attenuation, provided the attenuation (in dB/cm) is proportional to the frequency.

$$P(f_x, f_y, f | z_2) = P(f_x, f_y, f | z_1) \exp\left\{-i K(\vec{f})(z_2 - z_1)\right\} \exp(-\alpha f (z_2 - z_1))$$

$$= P(f_x, f_y, f | z_1) \exp\left\{-i K_a(\vec{f})(z_2 - z_1)\right\} \quad (20)$$

$$K_a(\vec{f}) \equiv K(\vec{f}) - i\alpha f$$

This gives the following modification of equation (11)

$$Hp_a(\vec{f}, \vec{f}') = Hp(\vec{f}, \vec{f}') \exp\left(\alpha f \frac{z_2}{2}\right) \cdot \exp\left(-\alpha f \left\{z_1 - \frac{z_2}{2}\right\}\right) = Hp(\vec{f}, \vec{f}') \exp(-\alpha f (z_1 - z_2)) \quad (21)$$

The fundamental wave in $z=z_1$, with attenuation is

$$P1_a(f_x, f_y, f) \equiv P1(f_x, f_y, f) \exp(-\alpha f z_1) \quad (22)$$

Combination of (11), (19), and (20) gives

$$P2_a(f_x, f_y, f) \equiv P2(f_x, f_y, f) \exp(-\alpha f z_2) \quad (23)$$

This shows that frequency dependent attenuation has the same effect to the second harmonic wave as in the linear case. This observation was also made by B. Angelsen in [Ultrasound Imaging, ch. 12?]

Two-way (transmit-receive response)

The transmit-receive response of the ultrasound imaging system can be found by convolution along the z -axis of the transmit and receive responses

$$h(x, y, z) = h_{Tx} \otimes_z h_{Rx}(z, y, z) \quad (24)$$

The convolution in z corresponds to multiplication in the f -domain. Since both transmit and receive responses are calculated in Fourier-domain (f_x, f_y, f) , the most computer-efficient way to obtain the 2-way response is by 2D Fouriertransform (f_x, f_y) of H_{Tx} , H_{Rx} , followed by multiplication. The space/time response can then be obtained by inverse fft along the f -dimension. The two-D (x, y) power profile can also be obtained directly by the square sum along the f -dimension.

In figure 2 examples of 2-way response for a dynamic receive aperture (fixed focus in elevation) is shown.

Implementation considerations

A first prototype of a simulation program "Propose" has been implemented in Matlab v6.1.

The spatial (x, y) and temporal (t) extent of the simulated beam is given by the quantisation step in f_x, f_y , and f , respectively. For simplicity, the same quantisation steps was used also in the calculation of the trippel-integral in (11). The linear wave $P1$, as well as the "katet" function $K(f_x, f_y, f)$ equ.(11) are tabulated in 3D array's. For apertures which are symmetrical in x and y , (like the rectangular aperture), the integral can be evaluated for only one quadrant in the (f_x, f_y) -plane, which reduces the cpu-time to 25%.

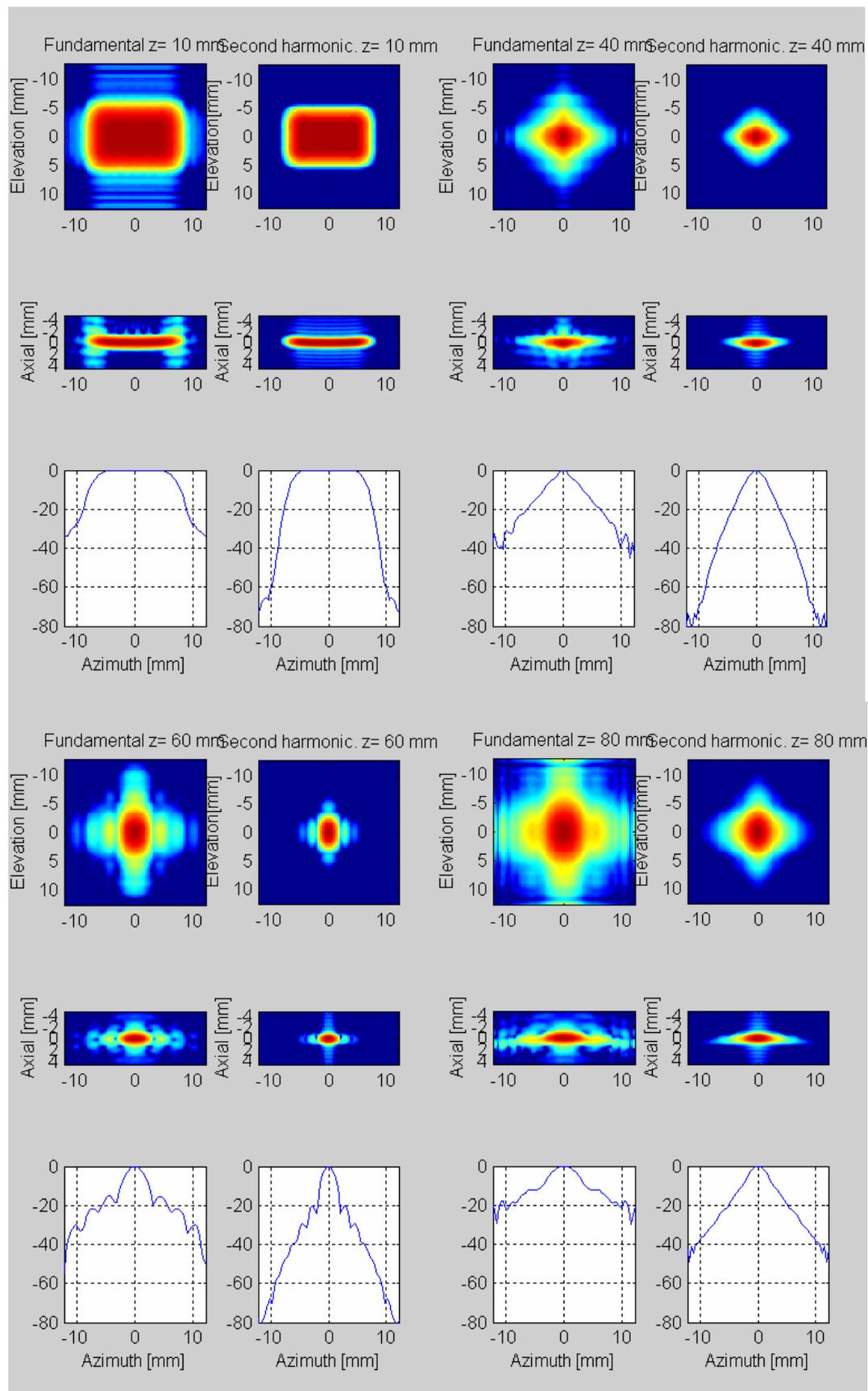


Figure 1. Fundamental and second harmonic field for $z=10, 40, 60$ and 80 mm. Dynamic range: 40 dB

Transducer parameters:

```
f0=1.67e6;  
B=1.0e6;%pulse bandwidth transmitted pulse  
c=1540;  
lambda=c/f0;Dx=18.6e-3;%aperure azimuth  
Dy=13e-3;%aperure elevation  
R=60e-3;%focal distance
```

```
Simulation range (x,y,z)= 25,25,and 10 mm  
Simulation time: 24 sec. ( PIII, 1.2 GHz)
```

Observations:

The side lobe suppression in the second harmonic beamprofile seems to be fairly close to the double (in dB) of the fundamental wave. In a region before the focal plane, the far sidelobes seems to be even lower than this.

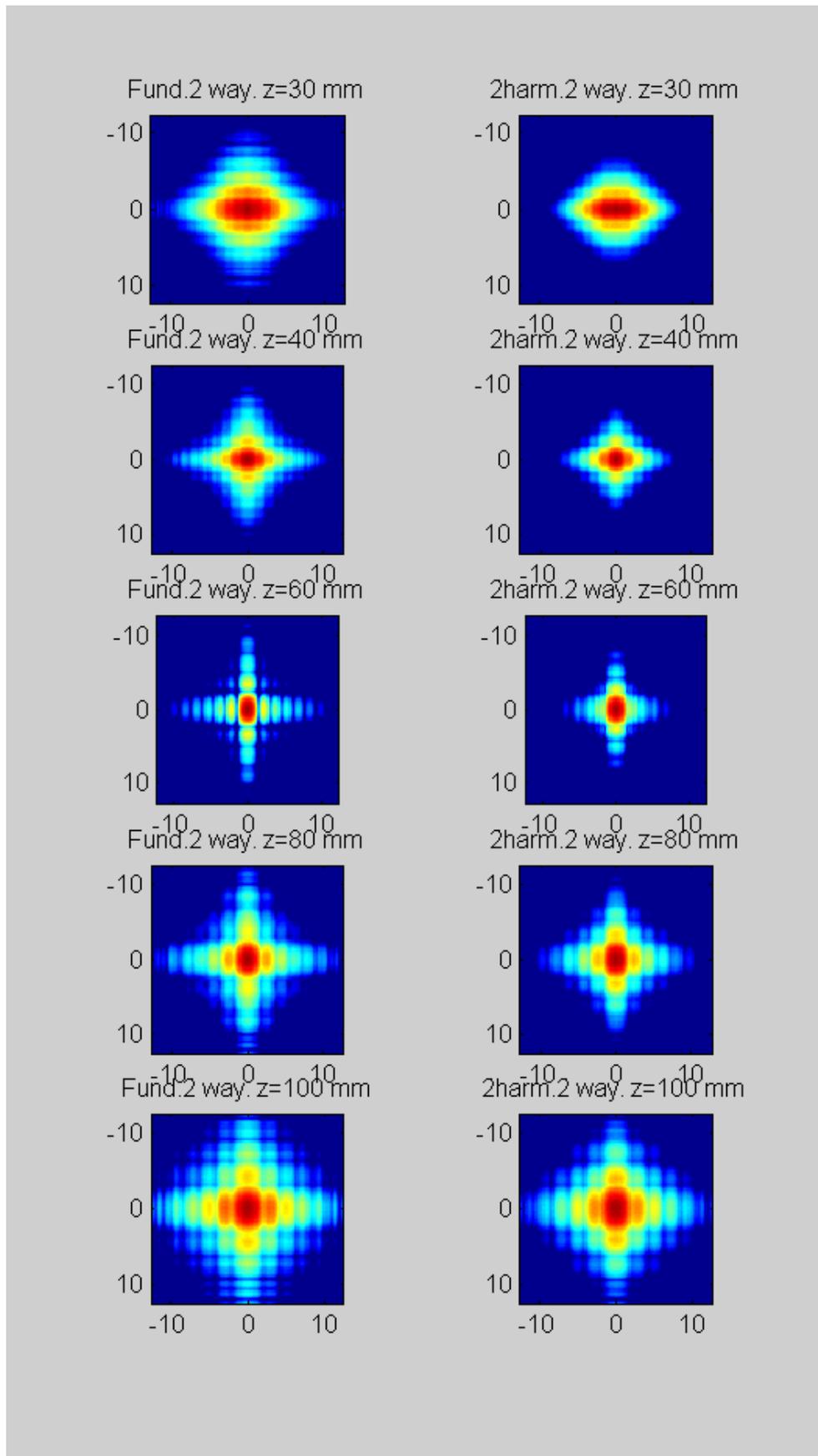


Figure 2. Two-way response. Same Tx aperture as in fig. 1 is also used for receive, with dynamic focusing in azimuth directio. Dynamic range 80 dB